ON NON-NEWTONIAN FLUID FLOWS UNDER CONDITIONS OF AN INHOMOGENEOUS TEMPERATURE DISTRIBUTION

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Variational principles for an incompressible viscous medium determining the velocity field in steady nonisothermal motion are formulated. The flow of a polyethylene melt is investigated under conditions of an inhomogeneous temperature distribution.

Polymer materials in a viscous-fluid state manifest the properties of a non-Newtonian fluid. Processes of fabricating polymers into products are associated with the flow of melts of these materials under conditions of an inhomogeneous temperature distribution.

Let us examine a nonisothermal non-Newtonian fluid in an arbitrary cylindrical channel. Let us introduce a fixed Cartesian coordinate system such that the z axis coincides with the channel axis. We assume the flow to be laminar, steady and directed along the channel axis. We also assume that the temperature T of the medium depends only on the x, y coordinates, where we consider the function T(x, y) known.* It can then be assumed that the velocity vector components are determined by

$$v_z = v(x,\ y), \quad v_x = v_y = 0. \tag{1}$$

Let us examine the flow of a nonlinearly viscous fluid, in which the stress tensor is determined to the accuracy of the hydrostatic pressure by the velocity gradients and temperature values in the time under consideration. The equation of state of such a fluid is represented in a sufficiently general case in the form

$$\sigma_{ij} = -P\delta_{ij} + \psi B_{ij}^{(1)}. \tag{2}$$

Here ψ is a scalar function dependent on the temperature and the second invariant of the strain rate tensor, i.e.,

$$\psi = \psi\left(I_2,\ T(x,\ y)\right), \tag{3}$$

where $I_2 = B_{ij}^{(1)} B_{ji}^{(1)}/2$.

The strain rate tensor components are

$$B_{ij}^{(1)} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \tag{4}$$

and in the case of the flow under consideration

$$I_2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2. \tag{5}$$

It can be shown that the relationship (2) admits the existence of a potential function $\Phi = \Phi(I_2, T(x, y))$ such that

* By virtue of the low heat conductivity of polymers, and the comparatively short length of the channels in machines fabricating these materials, the temperature of polymer melts varies quite inessentially along the channel length (see [4, 7], say). Hence, the assumption that the temperature of the medium depends only on the x and y coordinates is acceptable in application to the processes of polymer fabrication.

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$$\sigma_{ij} + P\delta_{ij} = s_{ij} = \frac{\partial \Phi}{\partial B_{ij}^{(1)}} . \tag{6}$$

Comparing the relationship

$$\frac{\partial \Phi}{\partial B_{ij}^{(1)}} = \frac{\partial \Phi}{\partial I_2} B_{ij}^{(1)} = s_{ij} \tag{7}$$

with (2), we find

$$\psi = \frac{\partial \Phi}{\partial I_2} . \tag{8}$$

Starting from (8), we have

$$\Phi = \int_0^{I_2} \psi(\xi, T(x, y))\, d\xi + C, \tag{9}$$

where C is an arbitrary constant; in particular, we can put C = 0.

Variational Principles. Variational principles can be formulated for a nonisothermal non-Newtonian fluid flow by comparing the steady flow state with the velocities v_i and an adjacent state with the velocities $v_i + \delta v_i$, where the possible velocities δv_i are combined with the incompressibility condition, but not absolutely with the equation of motion in stresses.

Let us expand the potential function in a Taylor series

$$\Phi (B_{ij}^{(1)} + \delta B_{ij}^{(1)}) = \Phi (B_{ij}^{(1)}) + \frac{\partial \Phi}{\partial B_{ij}^{(1)}} \delta B_{ij}^{(1)} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial B_{ij}^{(1)} \partial B_{kl}^{(1)}} \delta B_{ij}^{(1)} \delta B_{kl}^{(1)} + \ldots \tag{10}$$

Integrating (10) over the volume V taking account of (6), we arrive at

$$\int_V s_{ij} \delta B_{ij}^{(1)} = \Delta \int_V \Phi (B_{ij}^{(1)})\, dV - \int_V \Phi (\delta B_{ij}^{(1)})\, dV. \tag{11}$$

Using the law of conservation of mechanical energy, we arrive at the variational principle for an incompressible viscous medium governing the velocity field in steady nonisothermal flow:

$$\Delta\Big[\int_V \Phi (B_{ij}^{(1)})\, dV - 2 \int_S N_i v_i dS - 2 \int_V X_i v_i dV\Big] = \Delta F = 0 + \delta^2 \int_V \Phi (B_{ij}^{(1)})\, dV. \tag{12}$$

If the second variation $\delta^2 \int_V \Phi(B_{ij}^{(1)}) dV$ is positive, then the real velocity field determines the minimum of the functional F.

Neglecting volume forces, and taking account of (9), the functional F becomes

$$F = \int_V dV \int_0^{I_2} \psi (\xi, T(x, y))\, d\xi - 2 \int_S N_i v_i dS. \tag{13}$$

The following can be shown

$$N_z = \frac{\partial P}{\partial z} = \text{const} = -\frac{P_0 - P_1}{l}; \quad N_x = N_y = 0, \tag{14}$$

for a nonisothermal fluid flow in an arbitrary cylindrical channel when the velocity field is defined by (1) by starting from the equation of motion, where P_0 and P_1 are the hydrostatic pressure in the z = 0 and z = l sections, respectively.

Considering the part of the fluid between two cylinder cross-sections separated by a unit length at this time, and taking account of fluid adhesion to the cylinder surface and (14), we find the following for the functional (13):

$$F(v) = \int_\Omega d\Omega \int_0^{(I_2)} \psi (\xi, T(x, y))\, d\xi + 2 \frac{\partial P}{\partial z} \int_\Omega v d\Omega. \tag{15}$$

Let us solve the problem of the minimum of the functional (15) under the assumption that the function v satisfies the homogeneous boundary condition

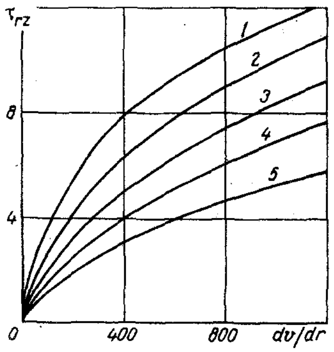

Fig. 1. Polyethylene flow curves at different temperatures (τ, 10^4 N/m^2, dv/dr, sec^{-1}): 1) T = 443°K; 2) 463°K; 3) 483°K; 4) 503°K; 5) 523°K.

$$v = 0 \text{ on } L. \tag{16}$$

Following Langenbach–Mikhlin [6], it can be proved that for a finite domain Ω bounded by a piecewise-smooth contour, and for a function $\psi(I_2, T(x, y))$ having continuous first-and second-order partial derivatives with respect to I_2 and T satisfying the inequalities:

$$\psi(I_2, T(x, y)) \geqslant C_1 \quad (C_1 = \text{const} > 0); \tag{17}$$

$$\psi(I_2, T(x, y)) + 2\frac{\partial\psi(I_2, T(xy))}{\partial I_2} I_2 \geqslant C_2, \quad (C_2 = \text{const} > 0), \tag{18}$$

the functional (15) in the space $W_2^{(1)}(\Omega)$ has an absolute minimum which is achieved at a single point. If v_0 is this point, then any minimizing sequence will converge to v_0 in the metric of the space $W_2^{(1)}(\Omega)$.

Let us examine the problem of nonisothermal polymer flow in a channel of elliptical cross-section.

As has been shown above, the problem of non-Newtonian fluid flow in cylindrical channels reduces to the problem of the minimum of the functional (15). The domain of the section Ω is an ellipse with the axes 2a and 2b. We consider the temperature field symmetric relative to the x and y axes.

Let us seek the function realizing an extremum of the functional (15) in the form

$$v_n = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)(A_1 + A_2x^2 + A_3y^2 + \ldots + A_nx^{2i}y^{2j}), \tag{19}$$

because of symmetry.

The elements of the coordinate system are hence linearly independent, form a complete system of functions, and satisfy the boundary condition of the problem. The parameters A_n are found from the condition of the minimum of the functional F(v) by using the Ritz method

$$\frac{\partial F(v)}{\partial A_m} = 2\iint_\Omega \psi(I_2, T(x, y))\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial A_m} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial A_m}\right)dxdy + 2\frac{\partial P}{\partial z}\iint_\Omega \frac{\partial v}{\partial A_m}dxdy = 0 \quad (m = 1, 2, \ldots, n). \tag{20}$$

The sequence $\{v_n\}$ thus constructed will be minimizing.

Let us investigate the flow of a polyethylene melt. The dependence found experimentally [1] between the shear stresses and the corresponding shear velocities for this material at the temperatures T_0 = 443°K, T_1 = 463°K, T_2 = 483°K, T_3 = 503°K, and T_4 = 523°K is represented by curves 1-5, respectively, in Fig. 1. We approximate these curves by the functions $\tau_{rz} = \tau_{rz}(\theta, dv/dr)$ continuous in the domain $0 \le \theta \le 80°$, $0 \le dv/dr < \infty$, where $\theta = T - T_0$, $T_0 \le T$. We give this function in the form

$$\tau_{rz}\left(\theta, \frac{dv}{dr}\right) = \begin{cases} \tau_{rz}^{(1)} = a_0(\theta)\dfrac{dv}{dr} + a_1(\theta)\left(\dfrac{dv}{dr}\right)^2 + a_2(\theta)\left(\dfrac{dv}{dr}\right)^3 + a_3(\theta)\left(\dfrac{dv}{dr}\right)^4 & \text{for } \dfrac{dv}{dr} \leqslant 600, \quad (21) \\ \tau_{rz}^{(2)} = b_0(\theta) + b_1(\theta)\dfrac{dv}{dr} & \text{for } \dfrac{dv}{dr} \geqslant 600. \quad (22) \end{cases}$$

Here the dimensionality of dv/dr is sec^{-1}, and of τ_{rz} is N/m^2.

The functions $a_0(\theta)$, $a_1(\theta)$, . . . , $a_3(\theta)$ are defined thus.

Values of $a_i(\theta_k)$ (i = 0, 1, 2, 3; k = 0, 1, 2, 3, 4), where $\theta_k = T_k - T_0$, were determined separately by the dependence (21) from the approximation of each curve in Fig. 1. By means of the values $a_i(\theta_k)$, the functions $a_i(\theta)$ were determined for which the following dependences had been obtained:

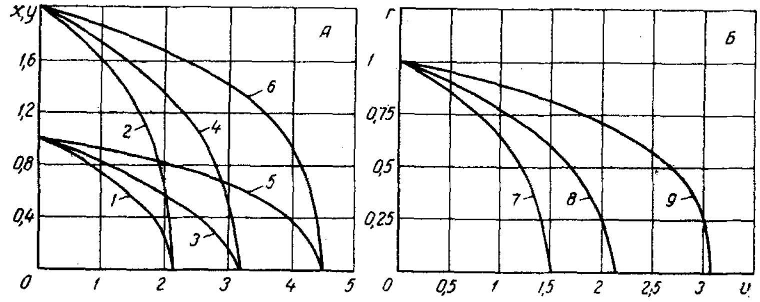

Fig. 2. Curves of the velocity distribution in a channel of elliptical cross-section A and circular section B (a = 2b, b = 1 cm) for different heat modes (x, y, r, cm; v, m/sec; 1, 2, 7) isothermal flow (T = 443°K); 3, 4, 8) polymer cooling (T(a, 0) = 443°K, T(0, b) = 458°K, T(0, 0) = 483°K); 5, 6, 9) polymer ating (T(a, 0) = 483°K, T(0, b) = 498°K, T(0, 0) = 443°K).

$$\begin{aligned} a_0(\theta) &= (0.46302\cdot 10^{-1} - 0.64664\cdot 10^{-3}\theta + 0.24408\cdot 10^{-5}\theta^2)\ \mathrm{N\cdot sec/m^2};\\ a_1(\theta) &= (-0.12762\cdot 10^{-3} + 0.21454\cdot 10^{-5}\theta - 0.73399\cdot 10^{-8}\theta^2)\ \mathrm{N\cdot sec^2/m^2};\\ a_2(\theta) &= (0.20441\cdot 10^{-6} - 0.35944\cdot 10^{-8}\theta + 0.10099\cdot 10^{-10}\theta^2)\ \mathrm{N\cdot sec^3/m^2};\\ a_3(\theta) &= (-0.12854\cdot 10^{-9} + 0.24183\cdot 10^{-11}\theta - 0.66632\cdot 10^{-14}\theta^2)\ \mathrm{N\cdot sec^4/m^2}. \end{aligned} \tag{23}$$

The approximations were made by the method of equal areas [5]. The functions $b_0(\theta)$, $b_1(\theta)$ were represented as

$$\begin{aligned} b_0(\theta) &= g_0 + g_1\theta + g_2\theta^2;\\ b_1(\theta) &= k_0 + k_1\theta + k_2\theta^2. \end{aligned} \tag{24}$$

We use the condition of continuity of the function $\tau_{rz}(\theta,\ dv/dr)$ to determine the coefficients g_i, k_i (i = 0, 1, 2):

$$\tau_{rz}^{(1)}\Big|_{\frac{dv}{dr}=600} = \tau_{rz}^{(2)}\Big|_{\frac{dv}{dr}=600} \tag{25}$$

Substituting (21), (22), into (25) and equating coefficients of equal powers of θ on the right and left sides, we obtain three equations to determine g_i and k_i.

We compose the missing three equations from the condition of equality of the values of τ_{rz} in the experimental and theoretical curves for dv/dr = 1000 and θ = 0, 40, 80. The following expressions were finally obtained:

$$\begin{aligned} b_0(\theta) &= (6.18 - 0.82328\cdot 10^{-1}\theta + 0.38309\cdot 10^{-3}\theta^2)\ \mathrm{N/m^2},\\ b_1(\theta) &= (0.525\cdot 10^{-2} + 0.62034\cdot 10^{-5}\theta - 0.40496\cdot 10^{-6}\theta^2)\ \mathrm{N\cdot sec/m^2}. \end{aligned} \tag{26}$$

The curves finally constructed by means of (21), (22), (23), and (26) coincide with the experimental curves within the limits of the accuracy of construction.

Taking (21) and (22) into account, we have

$$\psi = \begin{cases} \psi_1 = a_0(\theta) + a_1(\theta) I_2^{1/2} + a_2(\theta) I_2 + a_3(\theta) I_2^{3/2} & \text{for } \sqrt{I_2} \leqslant 600,\\ \psi_2 = b_0(\theta) I_2^{-1/2} + b_1(\theta) & \text{for } \sqrt{I_2} \geqslant 600. \end{cases} \tag{27}$$

Let us note that the representation of the function $\tau_{rz}(\theta,\ dv/dr)$ in the form (21) and (22) assures high accuracy of the approximation to the dependences found experimentally, and compliance with the inequality (18) at all points of the domain $0 \le dv/dr < \infty$; $0 \le \theta \le 80$.

Let us consider the temperature field in the medium to be symmetric relative to the x, y axes, and to be determined in the domain $\Delta(0 \le r \le 1;\ 0 \le \varphi \le \pi/2)$ by the expression

$$\theta = (1-r)\sum_{i=0}^{m} C_i r^i + \sum_{k=1}^{S} l_k r^{g(k)}\varphi^{j(k)} + l_0 \quad \begin{matrix}(j = 0;\ 1;\ 2\ \ldots),\\ (g = 1;\ 2;\ 3\ \ldots),\end{matrix} \tag{28}$$

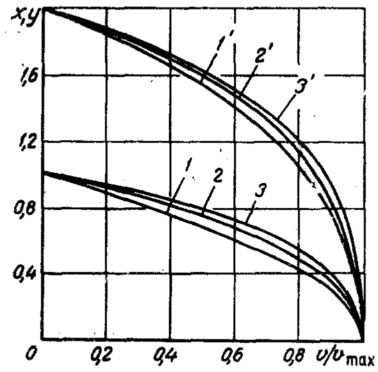

Fig. 3. Velocity profiles in a channel of elliptical section (x, y, in cm): 1, 1') polymer is cooled; 2, 2') isothermal flow; 3, 3') polymer is heated.

where C_i, l_k are constants, and r and φ are generalized polar coordinates related to the Cartesian x, y coordinates by the dependence

$$x = ar\cos\varphi, \quad y = br\sin\varphi. \tag{29}$$

The system (20) is solved by an iteration method [3]. The coefficients of the above-mentioned system were determined by integration over an elliptical domain. The integration was executed by means of quadrature formulas associated with the domain of integration [2, 3]. All the calculations were carried out on the BÉSM-2M.

An assertion on the existence and uniqueness of the solution for the variational problem for the functional (15) under the condition of the existence of continuous first and second partial derivatives of the function ψ was made above; however, the functions $\partial\psi/\partial I_2$ and $\partial^2\psi/\partial I_2^2$ become discontinuous at $\sqrt{I_2} = 600$ in the approximation mentioned. But this circumstance is not essential since the integrals in (20) were evaluated for values of the integrands at separate points.

Results of computations for the temperature field defined by the dependence

$$\theta = l_0 + (l_1 + l_2\varphi)\,r\,, \tag{30}$$

are presented in Figs. 2A, B.

The curves in Figs. 2A, B characterize the results of calculating the velocity distribution in a channel of elliptical section for $a = 2b$ (b = 1 cm) and of circular section ($-\partial P/\partial z = 13 \cdot 10^6$ N/m^3) for various heat modes.

On the average, the temperature field for the cases represented in Fig. 2B depends quite inessentially on φ, hence the velocity profiles in different directions agree within the limits of the accuracy of the construction.

The qualitative influence of the temperature distribution on the velocity profile is shown in Fig. 3 by curves of the dependence of v/v_{max} on x, y (v_{max} is the polymer velocity at the center), converted from the curves in Fig. 2A.

Let us note that the results shown in Fig. 3 can be explained from physical considerations. Where the polymer is relatively colder, its fluidity and velocity will diminish, while it will increase where it is warmer.

The results elucidated above have been obtained by retaining seven A_n coefficients in (19). A comparison between results of computations with seven and five parameters A_n showed that both results agree with 3-4% error in the velocity distribution at individual points, and up to 0.5% in the discharge.

Therefore, the velocity distribution in a nonlinearly viscous fluid moving in an arbitrary channel can be found by the proposed method when the temperature distribution in the medium is known. Let us note that the temperature field is unknown in the general case, and the problem reduces to the combined integration of the equations of motion and of energy balance. If such an integration is performed by successive approximations (i.e., by initially considering the velocity distribution the same as in an isothermal flow, a temperature field is found by which the velocity distribution is refined, the temperature field is determined by the found velocity distribution, etc.), then the successive approximation for the velocity field can be found by the proposed method.

NOTATION

x, y, z	are the Cartesian coordinates;
τ_{rz}	is the tangential shear stress;
dv/dx, dv/dy, dv/dr	are the shear velocity gradients;
T	is the temperature in the medium;
v_z	is the flow velocity of the medium;
σ_{ij}	are the stress tensor components;
P	is the hydrostatic pressure;
$B_{ij}^{(1)}$	are the strain rate tensor components;
ψ	is the apparent viscosity;
I_2	is the second invariant of the strain rate tensor;
Φ	is the potential function;
C, C_1, C_2	are the constants;
s_{ij}	is the stress tensor deviator;
F(v)	is the functional;
N_i, X_i	are the vector components of the external and volume forces;
S	is the surface of the medium under consideration;
Ω	is the cylinder cross-section;
L	is the boundary of the section Ω;
$\partial P/\partial z$	is the pressure drop per unit length;
P_0, P_1	are the hydrostatic pressure in the sections $z = 0$ and $z = l$;
a, b	are the semiaxes of the ellipse;
$W_2^{(1)}(\Omega)$	is the Sobolev space;
$\{v_n\}$	is the minimizing sequence;
θ	is the temperature in the medium;
r, φ	are the generalized polar coordinates.

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